F QUADRATIC EQUATIONS WITH $\Delta < 0$

Previously in this chapter, we determined that:

If $ax^2 + bx + c = 0$, $a \neq 0$ and $a, b, c \in \mathbb{R}$, then the **solutions** are $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$ is known as the **discriminant**.

We also observed that if $\Delta \geqslant 0$, the quadratic equation has real solutions.

It is in fact possible to find solutions for the case where $\Delta < 0$, but the solutions are not real numbers!

In 1572, Rafael Bombelli defined the **imaginary number** $i = \sqrt{-1}$. It is called 'imaginary' because we cannot place it on a number line. With i defined, we can write down solutions for quadratic equations with $\Delta < 0$. They are called *complex* solutions because they include a real and an imaginary part.

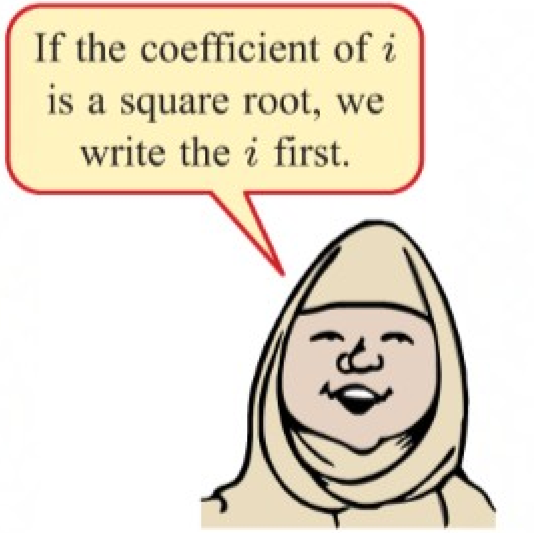
Any number of the form $a + bi$ where a and b are real and $i = \sqrt{-1}$, is called a **complex number**.

Example 16 Self Tutor

Write in terms of i:

a $\sqrt{-5}$

b $\sqrt{-16}$


If the coefficient of i is a square root, we write the i first.

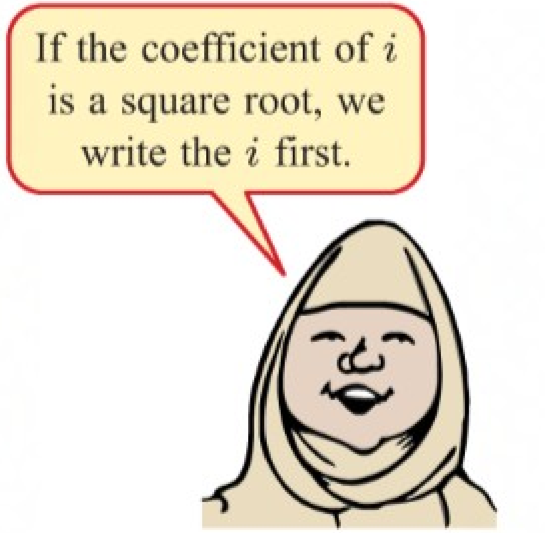
Example 16 **Self Tutor**

Write in terms of i:

a $\sqrt{-5}$ **b** $\sqrt{-16}$

a
$$\begin{aligned}\sqrt{-5} &= \sqrt{5\times -1}\\ &= \sqrt{5}\times\sqrt{-1}\\ &= i\sqrt{5}\end{aligned}$$

b
$$\begin{aligned}\sqrt{-16} &= \sqrt{16\times -1}\\ &= \sqrt{16}\times\sqrt{-1}\\ &= 4i\end{aligned}$$


If the coefficient of i is a square root, we write the i first.

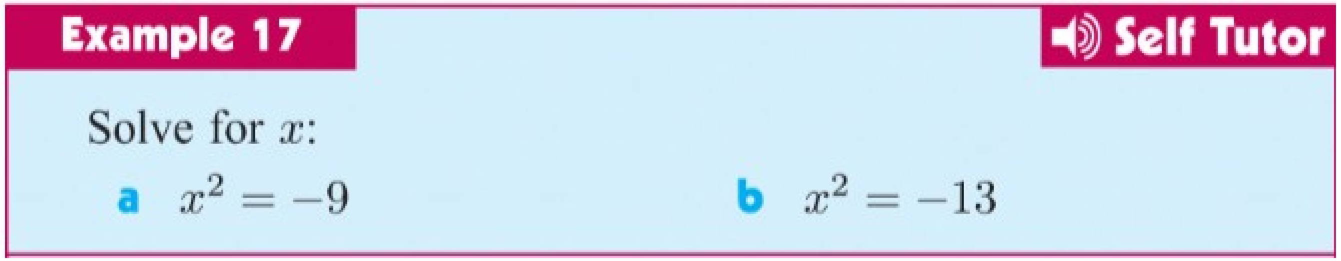
Example 17

Self Tutor

Solve for x:

a $x^2 = -9$

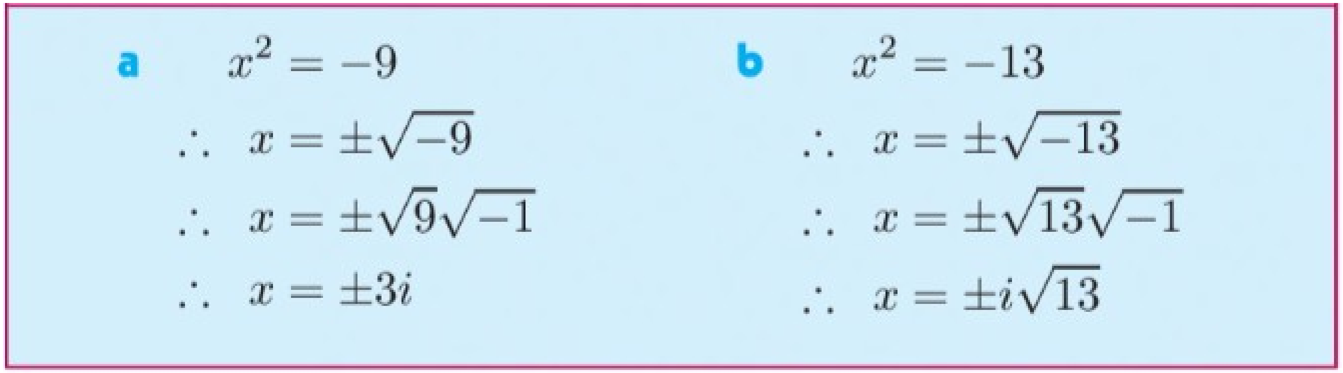
b $x^2 = -13$



Example 17 Self Tutor

Solve for x:

a $x^2 = -9$ **b** $x^2 = -13$



a $x^2 = -9$

$\therefore \quad x = \pm\sqrt{-9}$

$\therefore \quad x = \pm\sqrt{9}\sqrt{-1}$

$\therefore \quad x = \pm 3i$

b $x^2 = -13$

$\therefore \quad x = \pm\sqrt{-13}$

$\therefore \quad x = \pm\sqrt{13}\sqrt{-1}$

$\therefore \quad x = \pm i\sqrt{13}$

1 Write in terms of i:

a $\sqrt{-2}$ **b** $\sqrt{-25}$ **c** $\sqrt{-11}$ **d** $\sqrt{-81}$

2 Solve for x:

a $x^2 = -3$ **b** $x^2 = -4$ **c** $x^2 = -14$

d $x^2 + 7 = 0$ **e** $3x^2 = -48$ **f** $2x^2 + 15 = 3$

Example 18

Self Tutor

Solve for x:

a $x^2 + 2x + 2 = 0$

b $2x^2 - x + 3 = 0$

Example 18 **Self Tutor**

Solve for x:

a $x^2 + 2x + 2 = 0$

b $2x^2 - x + 3 = 0$

a $x^2 + 2x + 2 = 0$ has
$a = 1, \; b = 2, \; c = 2$

$$\therefore \; x = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$$

$$\therefore \; x = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$\therefore \; x = \frac{-2 \pm \sqrt{-4}}{2}$$

$$\therefore \; x = \frac{-2 \pm \sqrt{-1}\sqrt{4}}{2}$$

$$\therefore \; x = \frac{-2 \pm 2i}{2}$$

$$\therefore \; x = -1 \pm i$$

b $2x^2 - x + 3 = 0$ has
$a = 2, \; b = -1, \; c = 3$

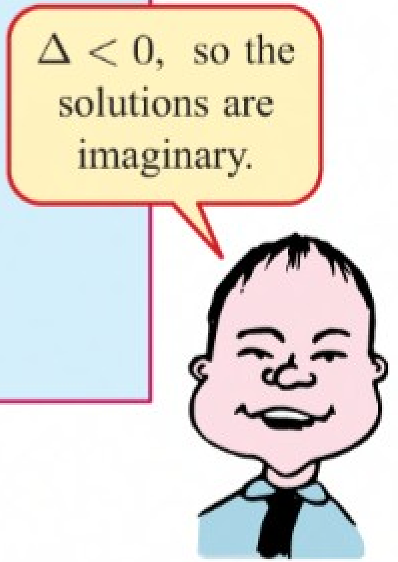
$$\therefore \; x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(3)}}{2(2)}$$

$$\therefore \; x = \frac{1 \pm \sqrt{1-24}}{4}$$

$$\therefore \; x = \frac{1 \pm \sqrt{-23}}{4}$$

$$\therefore \; x = \frac{1 \pm \sqrt{-1}\sqrt{23}}{4}$$

$$\therefore \; x = \frac{1 \pm i\sqrt{23}}{4}$$


$\Delta < 0$, so the solutions are imaginary.

3 Without solving them, decide whether these equations have two real solutions, one real solution, or imaginary solutions.

a $3x^2 + 5x - 1$

b $x^2 - 4x + 4 = 0$

c $2x^2 - 3x + 5 = 0$

4 Solve for x:

a $x^2 + x + 2 = 0$

b $x^2 - 3x + 6 = 0$

c $x^2 + 4x + 13 = 0$

d $2x^2 - x + 5 = 0$

e $-x^2 + 2x - 17 = 0$

f $3x^2 + 5 = 3x$

1 **a** $i\sqrt{2}$ **b** $5i$ **c** $i\sqrt{11}$ **d** $9i$

2 **a** $x = \pm i\sqrt{3}$ **b** $x = \pm 2i$ **c** $x = \pm i\sqrt{14}$

d $x = \pm i\sqrt{7}$ **e** $x = \pm 4i$ **f** $x = \pm i\sqrt{6}$

3 **a** two real solutions **b** one real solution

c imaginary solutions

4 **a** $x = \dfrac{-1 \pm i\sqrt{7}}{2}$ **b** $x = \dfrac{3 \pm i\sqrt{15}}{2}$ **c** $x = -2 \pm 3i$

d $x = \dfrac{1 \pm i\sqrt{39}}{4}$ **e** $x = 1 \pm 4i$ **f** $x = \dfrac{3 \pm i\sqrt{51}}{6}$